Earth azimuth effect in the bank of search templates for an all sky search of the continuous gravitational wave

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Abstract

We study the problem of all sky search in reference to continuous gravitational wave (CGW) whose wave-form are known in advance. We employ the concept of Fitting Factor and study the variation in the bank of search templates with different Earth azimuth at $t = 0$. We found that the number of search templates varies significantly. Hence, accordingly, the computational demand for the search may be reduced up to two orders by time shifting the data. **Keywords:** gravitational wave, data analysis, all sky search, LIGO.

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1 Introduction

The ground based laser interferometric gravitational wave (GW) detectors viz. LIGO[1], VIRGO[2], GEO600[3] TAMA 300[4] produce a single data stream that may contain continuous, chirp, burst and stochastic GW. These detectors don't point, but rather sweep their broad quadrupolar beam pattern across the sky as the earth moves. Hence the data analysis system will have to carry out all sky searches for its sources. In this, the search of continuous gravitational wave (CGW) without a priori knowledge appears to be computationally quite demanding even by the standard computers expected to be available in the near future. It appears that due to limited computational resource it will be not feasible to do all sky all frequency search for the CGW in the months/year data set. Hence for the search, one have to look for the signal in the short duration of data set. Understanding the problem and the target sensitivity of the advance LIGO, it may be feasible to do all sky search for one day data set at low and in narrow frequency band. The search may be more significant if it is done in the frequency band where most of the Pulsars are detected by other means. Also, the choice of optimal data processing and a clever programming is also integral part of a solution to this problem. Amongst these the pre-correction of time series due to the Doppler modulation before the data is processed may be a method, which will reduce the computational requirements. In reference to this, Schutz[5] has introduced the concept of patch in the sky as the region of space throughout which the required Doppler correction remains the same. He shown that the number of patches required for 10^7 sec. observation data set of one KHz signal would be about 1.3×10^{13}. This also implies that the bank of search templates require for the match filtering in an all sky search. However, the size of the patch would also depend on the data analysis technique being employed, which in turn depend on the parameters contain in the phase of the modulated signal. Hence, in this paper after incorporating the azimuth of the Earth (β_{orb}) at $t = 0$ in the Fourier transform (FT) obtained by Srivastava and Sahay[6], we investigate its effect in the bank of search templates for one sidereal data set. Hence in the next section we extend the FT of the Frequency modulated (FM) CGW obtained by them[6] with taking account of β_{orb}. In section 3 we employ the concept of Fitting Factor (FF)[7] and check the cross correlation of the templates with the corresponding data set exceeds the preassigned threshold by considering

the source location as parameters at different β_{orb} of the signal manifold and compute the number of search templates require for an all sky search applicable to such analysis. We present our conclusion in section 4.

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2 Fourier transform of the Frequency modulated continuous gravitational wave

The FT analysis of the Frequency modulated CGW has been done by Srivastava and Sahay[6] by taking account the effects arising due to the rotational as well as orbital motion of the Earth. However, they have neglected an important parameter β_{orb}. To obtain FT by taking account of β_{orb}, we rewrite the phase of the received CGW signal of frequency f_o at time t with some modification given by them[6] and may be written as

$$\Phi(t) = 2\pi f_o t + \mathcal{Z}\cos(a\xi_{rot} - \sigma) + \mathcal{N}\cos(\xi_{rot} - \delta) - \mathcal{M} \tag{1}$$

where

$$\left.\begin{aligned}
\mathcal{M} &= \tfrac{2\pi f_o}{c}\left(R_{se}\sin\theta\cos\sigma + \sqrt{\mathcal{P}^2+\mathcal{Q}^2}\cos\delta\right), \\
\mathcal{Z} &= \tfrac{2\pi f_o}{c}R_{se}\sin\theta\,, \quad \mathcal{N} \;=\; \tfrac{2\pi f_o}{c}\sqrt{\mathcal{P}^2+\mathcal{Q}^2}\,, \\
\mathcal{P} &= R_e\sin\alpha(\sin\theta\sin\phi\cos\epsilon + \cos\theta\sin\epsilon)\,, \\
\mathcal{Q} &= R_e\sin\alpha\sin\theta\cos\phi\,, \\
\sigma &= \phi - \beta_{orb}\,, \quad \delta = \tan^{-1}\tfrac{\mathcal{P}}{\mathcal{Q}} - \beta_{rot}\,, \\
a &= w_{orb}/w_{rot} \;\approx\; 1/365.26, \quad w_{orb}t \;=\; a\xi_{rot}, \\
\mathbf{n} &= (\sin\theta\cos\phi,\; \sin\theta\sin\phi\,,\; \cos\theta)\,, \quad \xi_{rot} \;=\; w_{rot}t
\end{aligned}\right\} \tag{2}$$

where θ, ϕ, R_e, R_{se}, w_{orb} w_{rot}, α and ϵ represent respectively the celestial co-latitude, longitude, Earth radius, average distance between Earth centre from the origin of SSB frame,

orbital and rotational angular velocity of the Earth, co-latitude of the detector and obliquity of the ecliptic. Here β_{orb} and β_{rot} are the azimuth of the Earth and detector at $t = 0$ respectively.

Now, the two polarisation functions of CGW can be described as

$$h_+(t) = h_{o_+} \cos[\Phi(t)] \tag{3}$$

$$h_\times(t) = h_{o_\times} \sin[\Phi(t)] \tag{4}$$

where h_{o_+}, $h_{o_\times}$ are constant amplitude of the two polarizations.

Considering the function

$$h(t) = \cos[\Phi(t)] \tag{5}$$

the FT for one sidereal day may be given as

$$\left[\tilde{h}(f)\right]_d = \int_0^{T_{obs}} \cos[\Phi(t)] e^{-i2\pi ft} dt \; ; \qquad T_{obs} = one\ sidereal\ day = 86164\ s \tag{6}$$

which may be split into

$$\left[\tilde{h}(f)\right]_d = I_{\nu_-} + I_{\nu_+} \; ; \tag{7}$$

$$I_{\nu_-} = \frac{1}{2w_{rot}} \int_0^{2\pi} e^{i[\xi\nu_- + \mathcal{Z}\cos(a\xi - \sigma) + \mathcal{N}\cos(\xi - \delta) - \mathcal{M}]} d\xi \, , \tag{8}$$

$$I_{\nu_+} = \frac{1}{2w_{rot}} \int_0^{2\pi} e^{-i[\xi\nu_+ + \mathcal{Z}\cos(a\xi - \sigma) + \mathcal{N}\cos(\xi - \delta) - \mathcal{M}]} d\xi \, , \tag{9}$$

$$\nu_\pm = \frac{f_o \pm f}{f_{rot}}; \quad \xi = \xi_{rot} = w_{rot} t \tag{10}$$

As I_{ν_+} contributes very little to $\left[\tilde{h}(f)\right]_d$. Hence, hereafter, we drop I_{ν_+} from Eq. (7) and write ν in place of ν_-. Using the identity

$$e^{\pm i\kappa\cos\vartheta} = J_o(\pm\kappa) + 2\sum_{l=1}^{l=\infty} i^l J_l(\pm\kappa)\cos l\vartheta \tag{11}$$

we obtain

$$\left[\tilde{h}(f)\right]_d \simeq \frac{1}{2w_{rot}} e^{-i\mathcal{M}} \int_0^{2\pi} e^{i\nu\xi} \left[J_o(\mathcal{Z}) + 2\sum_{k=1}^{k=\infty} J_k(\mathcal{Z}) i^k \cos k(a\xi - \sigma)\right] \times \left[J_o(\mathcal{N}) + 2\sum_{m=1}^{m=\infty} J_m(\mathcal{N}) i^m \cos m(\xi - \delta)\right] d\xi \qquad (12)$$

where J stands for the Bessel function of the first kind. After integration we get

$$\left[\tilde{h}(f)\right]_d \simeq \frac{\nu}{2w_{rot}} \sum_{k=-\infty}^{k=\infty} \sum_{m=-\infty}^{m=\infty} e^{i\mathcal{A}} \mathcal{B}[\mathcal{C} - i\mathcal{D}] \; ; \qquad (13)$$

$$\left.\begin{array}{lcl}
\mathcal{A} & = & \frac{(k+m)\pi}{2} - \mathcal{M} \\
\mathcal{B} & = & \frac{J_k(\mathcal{Z}) J_m(\mathcal{N})}{\nu^2 - (ak+m)^2} \\
\mathcal{C} & = & \sin 2\nu\pi \cos(2ak\pi - k\sigma - m\delta) - \frac{ak+m}{\nu}\{\cos 2\nu\pi \sin(2ak\pi - k\sigma - m\delta) + \sin(k\sigma + m\delta)\} \\
\mathcal{D} & = & \cos 2\nu\pi \cos(2ak\pi - k\sigma - m\delta) + \frac{ka+m}{\nu} \sin 2\nu\pi \sin(2ak\pi - k\sigma - m\delta) - \cos(k\sigma + m\delta)
\end{array}\right\}$$

Now its straight forward to obtain the FT of the two polarisation states of the wave and can be written as

$$\left[\tilde{h}_+(f)\right]_d = h_{o_+} \left[\tilde{h}(f)\right]_d \simeq \frac{\nu h_{o_+}}{2w_{rot}} \sum_{k=-\infty}^{k=\infty} \sum_{m=-\infty}^{m=\infty} e^{i\mathcal{A}} \mathcal{B}[\mathcal{C} - i\mathcal{D}] \; ; \qquad (14)$$

$$\left[\tilde{h}_\times(f)\right]_d = -ih_{o_\times} \left[\tilde{h}(f)\right]_d \simeq \frac{\nu h_{o_\times}}{2w_{rot}} \sum_{k=-\infty}^{k=\infty} \sum_{m=-\infty}^{m=\infty} e^{i\mathcal{A}} \mathcal{B}[\mathcal{D} - i\mathcal{C}] \qquad (15)$$

For the analysis we reduce the computing time $\approx 50\%$ by using symmetrical property of the Bessel functions by rewriting the Eq. (13) as

$$
\begin{aligned}
\left[\tilde{h}(f)\right]_d \;\simeq\; & \frac{\nu}{w_{rot}}\left[\frac{J_o(\mathcal{Z})J_o(\mathcal{N})}{2\nu^2}\left[\{\sin\mathcal{M}-\sin(\mathcal{M}-2\nu\pi)\}\;+i\{\cos\mathcal{M}-\cos(\mathcal{M}-2\nu\pi)\}\right]\;+\right.\\
& J_o(\mathcal{Z})\sum_{m=1}^{m=\infty}\frac{J_m(\mathcal{N})}{\nu^2-m^2}\left[(\mathcal{Y}\mathcal{U}-\mathcal{X}\mathcal{V})-i(\mathcal{X}\mathcal{U}+\mathcal{Y}\mathcal{V})\right]\;+\\
& \left.\sum_{k=1}^{k=\infty}\sum_{m=-\infty}^{m=\infty}e^{i\mathcal{A}}\mathcal{B}\left(\tilde{\mathcal{C}}-i\tilde{\mathcal{D}}\right)\right]\;;
\end{aligned}
\tag{16}
$$

$$
\left.
\begin{aligned}
\mathcal{X} &= \sin(\mathcal{M}-m\pi/2)\\
\mathcal{Y} &= \cos(\mathcal{M}-m\pi/2)\\
\mathcal{U} &= \sin 2\nu\pi\cos m(2\pi-\delta)-\tfrac{m}{\nu}\{\cos 2\nu\pi\sin m(2\nu\pi-\delta)-\sin m\delta\}\\
\mathcal{V} &= \cos 2\nu\pi\cos m(2\pi-\delta)+\tfrac{m}{\nu}\sin 2\nu\pi\sin m(2\pi-\delta)-\cos m\delta
\end{aligned}
\right\}
\tag{17}
$$

3 Bank of search templates for an all sky search

The study of templates has been made by many research workers[8, 9, 10, 11]. However, the question of possible minimum efficient interpolated representation of the correlators for an all sky search is a problem of interest. In this, the study of the variation in the bank of search templates for the short duration of data set in reference to the parameter contain in the phase of modulated signal is very important. Hence we check the effect of β_{orb} in the modulated signal by plotting $\left[\tilde{h}(f)\right]_d$ (Figures (1) and (2)) for LIGO detector at Hanford (the position and orientation of the detector can be found in Ref. 12) of unit amplitude signal for

$$
\left.
\begin{array}{ll}
f_o = 50\ Hz\,, & \beta_o = 0, \pi/2\\
\theta = \pi/18\,, & \phi = \pi/4
\end{array}
\right\}
\tag{18}
$$

Here we take the ranges[13, 14] of k and m as 1 to 27300 and -10 to 10 respectively because the value of Bessel functions decreases rapidly as its order exceed the argument. From the plot, we observe the obvious but major shift in the spectrum with β_{orb}. Understanding the effect we check the variation in the bank of search templates at different β_{orb}.

The bank of search templates are discrete set of signals from among the continuum of possible signals. Consequently all the signals will not get detected with equal efficiency. However, it is possible to choose judiciously the set of templates so that all the signals of a given amplitude are detected with a given minimum detection loss. FF is one of the standard measure for deciding what class of wave form is good enough and quantitatively describes the closeness of the true signals to the template manifold in terms of the reduction of SNR arising due to the cross correlation of a signal outside the manifold with the best matching templates lying inside the manifold, given as

$$FF = \frac{\langle h(f)|h_T(f;\theta_T,\phi_T)\rangle}{\sqrt{\langle h_T(f;\theta_T,\phi_T)|h_T(f;\theta_T,\phi_T)\rangle\langle h(f)|h(f)\rangle}} \tag{19}$$

where $h(f)$ and $h_T(f;\theta_T,\phi_T)$ represent respectively the FTs of the actual signal wave form and the templates. The inner product of two waveform h_1 and h_2 is defined as

$$\begin{aligned}\langle h_1|h_2\rangle &= 2\int_0^\infty \frac{\tilde{h}_1^*(f)\tilde{h}_2(f)+\tilde{h}_1(f)\tilde{h}_2^*(f)}{S_n(f)}df \\ &= 4\int_0^\infty \frac{\tilde{h}_1^*(f)\tilde{h}_2(f)}{S_n(f)}df \end{aligned}\tag{20}$$

where * denotes complex conjugation, $\tilde{}$ denotes the FT of the quantity underneath. $S_n(f)$ is the spectral noise density of the detector and has been taken stationary and Gaussian as the bandwidth of the signal is extremely narrow.

To compute the number of search templates we consider the LIGO detector at Hanford, receive a CGW signal of frequency $f_o = 50$ Hz from a source located at $(\theta,\phi) = (0.1^o, 30^o)$. We chosen the data set such that $\beta_{orb} = 0$ at $t = 0$. In this case we take the ranges of k and m as 1 to 310 and -10 to 10 respectively and bandwidth equal to 2.0×10^{-3} Hz for the integration. Now, we select the spacing $\triangle\theta = 4.5\times10^{-5}$, thereafter we maximize over ϕ by introducing spacing $\triangle\phi$ in the so obtained bank of search templates and determine the resulting FF. In similar manner we obtain the FF at $\beta_{orb} = \pi/4$ and $\pi/2$. The results obtained are shown in the Fig. (3). We observe that the nature of the curve is similar. Hence, it may be interesting to obtain a best fit of the graphs and may be given as

$$N_{Templates} = 10^{15}[c_o + c_1x - c_2x^2 + c_3x^3 - c_4x^4 + c_5x^5 - c_6x^6 + c_7x^7]\ ; \tag{21}$$

$0.80 \leq x \leq 0.995$

where $c_o, c_1, c_2, c_3, c_4, c_5, c_6, c_7$ are the constants given in the Table (1).

In view of the above investigation, the grid spacing ($\triangle\theta, \triangle\phi$) in the $(\theta, \phi)-$parameter of templates may be expressed as

$$\triangle\theta = \mathcal{F}(FF, f_o, \theta, \phi, T_{obs}, \beta_{orb}) \tag{22}$$

Similarly,

$$\triangle\phi = \mathcal{G}(FF, f_o, \theta, \phi, T_{obs}, \beta_{orb}) \tag{23}$$

In Ref. to 15, for one sidereal data set, the dependence of *FF* on the template variables θ_T and ϕ_T is given as

$$FF = e^{-0.00788(\theta\ -\ \theta_T)^2} \tag{24}$$

$$FF = e^{-0.01778(\phi\ -\ \phi_T)^2} \tag{25}$$

From Eqs. (22), (23), (24) and (25), we may write

$$\mathcal{F}(FF, 50, 0.1^o, 30^o, 1d, \beta_{orb}) = \left[-(0.00788)^{-1}\ln(FF)\right]^{1/2} \tag{26}$$

$$\mathcal{G}(FF, 50, 0.1^o, 30^o, 1d, \beta_{orb}) = \left[-(0.01778)^{-1}\ln(FF)\right]^{1/2} \tag{27}$$

Hence, for the selected FF one can determine $\triangle\theta$ and $\triangle\phi$. However, there is no unique choice for it. Here we are interested in the assignment of $\triangle\theta$ and $\triangle\phi$ such that the spacing is maximum resulting into the least number of templates. As we have mentioned earlier, there is stringent requirement on reducing computer time. Accordingly, there is serious need of adopting some procedure/formalism to achieve this. For example, one may adopt the method of hierarchical search.

Table 1: Coefficients of the best fit graphs obtained for the bank of search templates.

β_{orb}	c_o	c_1	c_2	c_3	c_4	c_5	c_6	c_7
0^o	-4.36537	34.4523	116.44	218.462	245.734	165.719	62.0404	9.94642
45^o	-403.012	3187.44	10795.5	20296.6	22877.7	15459.9	5799.45	931.641
90^o	-263.622	2086.48	7071.80	13305.6	15008.9	10150.3	3810.65	612.645

4 Conclusions

We have incorporated the parameter β_{orb} in the FM signal and investigated its effect in the bank of search templates for an all sky search. The analysis for complete response of the detector has not been done as the requisite is analogous to what presented by them[6] and also not required for the templates analysis.

We observe that the change of β_{orb} affects the spectra severely. Consequently in the bank search templates. On investigation for one sidereal day data set of a signal of 50 Hz, we found that the number of search templates varies significantly with β_{orb}. For the case investigated we found that for $FF = 0.97$ approximately 24.8267×10^{10}, 22.7840×10^{12} and 32.3097×10^{12} search templates may be require, if $\beta_{orb} = 0, \pi/4$ and $\pi/2$ respectively. Hence, the computational demand may be reduce up to two orders by time shifting the data. One may optimize the require number of search templates by doing analysis with the method given by Owen[16].

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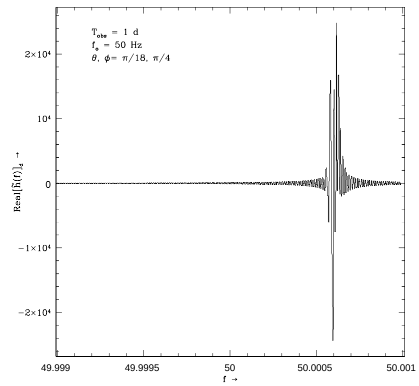

Figure 1: Spectrum of a Doppler modulated signal when $\beta_{orb} = 0$.

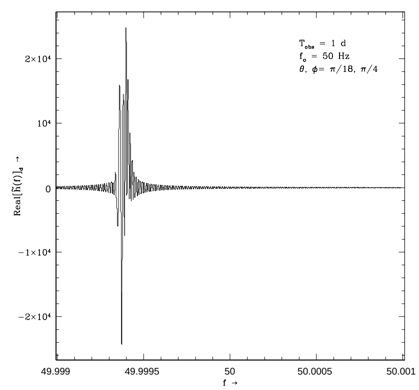

Figure 2: Spectrum of a Doppler modulated signal when $\beta_{orb} = \pi/2$.

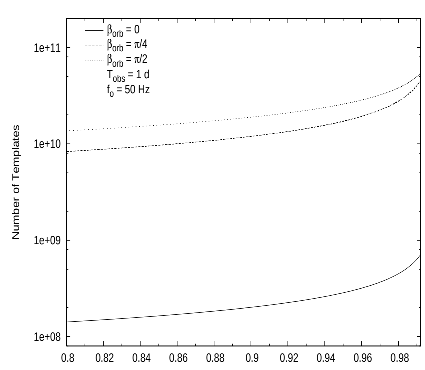

Figure 3: Variation in the number of search templates with FF at at different β_{orb}.